Midsemestral

Class Field Theory

Instructor: Ramdin Mawia **Time:** February 22, 2023; 10:00–13:00.

INSTRUCTIONS

i. Attempt FOUR problems, including problem n° 5. Each question carries 10 marks. The maximum you can score is **30**.

ii. You may use any of the results proved in class, unless you are asked to prove or justify the result itself. You may also use results from other problems in this question paper, provided you attempt and correctly solve the problem.

iii. The notation is standard: $\mathbb{A}_K$ denotes the ring of adèles of a global field K, and $\mathbb{A}_{K,S} = \{x \in \mathbb{A}_K : |x_v|_v \leqslant 1 \forall v \notin S\}$. We write $\mathbb{A}_{K,\infty} = \mathbb{A}_{K,\Sigma_\infty}$ where Σ_∞ is the set of archimedean places of K. Also, $\mathbb{A}_K^\times$ denotes the group of idèles and $\mathbb{A}_K^1$ denotes the group of 1-idèles (i.e., of content 1).

Global Fields

1. Let L/K be a finite Galois extension of number fields with Galois group G. Let v be a place of K and w be a place of L lying above v. Prove that the completions L_w and $L_{\sigma w}$ are isomorphic for any $\sigma \in G$.

2. Let K be a number field and S be a finite set of places of K, containing the archimedean places. Let $A_S := \{x \in K : |x|_v \leqslant 1 \forall v \notin S\}$. Prove that the quotient $A_S^\times \backslash \mathbb{A}_{K,S}^1$ is compact. Here $\mathbb{A}_{K,S}^1$ denotes the group of S-idèles of content 1.

3. Prove that $\mathcal{O}_K = K \cap \mathbb{A}_{K,\infty}$ for a number field K.

4. Let $\mathbb{A}_f$ denote the ring of finite adèles over the rationals, that is, the restricted direct product of the $\mathbb{Q}_p$ with respect to the $\mathbb{Z}_p$ for $p < \infty$, equipped with the restricted direct product topology (called the finite adèle topology). For each finite adèle $x = (x_2, x_3, x_5, \ldots) \in \mathbb{A}_f$, define $\|x\|_1 = \max_p |x_p|_p$ and $\|x\|_2 = \max_p |x_p|_p / p$. Show that $\|\cdot\|_1$ and $\|\cdot\|_2$ give metrics on $\mathbb{A}_f$.

   i. Prove that the topology induced by $\|\cdot\|_1$ is not the same as the finite adèle topology.
   ii. Prove that the topology induced by $\|\cdot\|_2$ is the same as the finite adèle topology.
   iii. Show that $\mathbb{A}_\mathbb{Q}$ is metrisable.

5. State true or false, with brief but complete justifications (**any five**):

   (a) There is a number field K such that the group of units $\mathcal{O}_K^\times$ of its ring of integers $\mathcal{O}_K$ is isomorphic to the unit circle $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$.
   (b) The field $\mathbb{R}((X))$ of Laurent series in one variable over $\mathbb{R}$ has a norm $\|\cdot\|$ (as an $\mathbb{R}$-vector space) such that $\|fg\| \leqslant \|f\|\|g\|$ for all $f, g \in \mathbb{R}((X))$. Here $\mathbb{R}$ is equipped with its usual absolute value.
   (c) For any number field K, we have $\mathcal{O}_K + \mathbb{A}_{K,\infty} = \mathbb{A}_K$.
   (d) The embedding $j : \mathbb{A}_K^\times \hookrightarrow \mathbb{A}_K \times \mathbb{A}_K$ given by $j(x) = (x, x^{-1})$ is continuous and is a homeomorphism on its image. Here $\mathbb{A}_K^\times$ is equipped with its usual idèle topology and $\mathbb{A}_K \times \mathbb{A}_K$ is equipped with the product topology.
   (e) For each place v of a number field K, there is a continuous surjection $\mathbb{A}_K \to K_v$.
   (f) If H is an open subgroup of a compact topological group G, then the index $[G : H]$ is finite.
   (g) If Γ is a discrete subgroup of (the additive group) $\mathbb{R}^n$, then Γ is free of finite rank.[1]

[1] See also Problem n° 3.